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Mixtures and Pure States in Relativistic Schrödinger Theory

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In relativistic Schrödinger theory, a physical system can be described by a wave function ($\rightsquigarrow$ "pure state") or by an intensity matrix ($\rightsquigarrow$ "mixture"). Since the space-time evolution of the system is described by a *non*-Hermitian Hamiltonian, transmutations of mixtures into pure states (and vice versa) would be formally possible. Nevertheless, the transition of a mixture into a pure state is dynamically forbidden, whereas the pure states are unstable and decay into mixtures. This effect is demonstrated by considering the Klein–Gordon–Higgs equations over an expanding Robertson–Walker universe.

1. INTRODUCTION

Perhaps one of the most universal equations of motion in physics is the Schrödinger equation, which is thought to govern the time development of any physical system. But what is the origin of the overwhelming success of such a relatively simple equation? In search of a better understanding of this success it seems reasonable to first have a closer look at the mathematical properties of the Schrödinger equation. For it appears very natural to assume that those mathematical properties are in intimate correspondence to the fundamental logic obeyed by nature in organizing the spatial and temporal evolution of matter. According to Schrödinger's original idea (Wheeler and Zurek, 1983), this evolution should occur in such a way that a physical system is characterized by some wave function ψ whose rate of change equals the action of the Hamiltonian operator $\hat{H}$ upon ψ,i.e., in modern notation,

$$i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle \tag{1.1}$$

Though the physical interpretation of this equation has been shifted somewhat

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off Schrödinger's original conception into the direction of a purely statistical approach, the general form (1.1) has survived all attempts at falsification and persists up to the present day.

What, then, is the crucial mathematical element inherent in (1.1) which so strikingly aims at the central point of natural phenomena? Surely, once the state vector $|\psi\rangle$ as an element of some Hilbert space has been adopted as the adequate concept for our mathematical modelling of natural processes, all further significance is carried by the Hamiltonian $\hat{H}$. Obviously, the properties of $\hat{H}$ uniquely determine how the state vector $|\psi\rangle$ moves around in Hilbert space, and it is just in this way that we are provided with the ability of predicting the future behavior of the physical system under consideration, albeit in a purely statistical sense. Now, one of the crucial properties of Schrödinger's Hamiltonian has been revealed to consist in its *Hermiticity* ($\hat{H} = \hat{H}^+$), which immediately implies certain *conservation laws*. Indeed, the validity of these conservation laws is stringent with regard to both the logical consistency of the theory and the experimental evidence. For instance, an immediate consequence of Hermiticity is the time independence of the norm of the state vector $|\psi\rangle$:

$$\frac{d}{dt}\langle\psi|\psi\rangle = 0 \tag{1.2}$$

i.e., the "conservation of probability." However, in view of the statistical variety of physical systems, their characterization by a single state vector $|\psi\rangle$ has been found to be too restrictive and one has to resort to the more general concept of the density matrix $\hat{\rho}$. Its equation of motion is readily deduced from Schrödinger's equation (1.1) for the wave function ψ as

$$i\hbar\frac{d\hat{\rho}}{dt} = [\hat{H}, \hat{\rho}] \tag{1.3}$$

The corresponding generalization of the conservation law (1.2) for "pure states" $|\psi\rangle$ to the larger class of statistical "mixtures" described by $\hat{\rho}$ is then found as

$$\frac{d}{dt}(\mathrm{tr}\,\hat{\rho}) = 0 \tag{1.4}$$

and the previous case of pure states $|\psi\rangle$, (1.2), is recovered as a subset of the mixtures $\hat{\rho}$, (1.4), by putting

$$\hat{\rho} \Rightarrow |\psi\rangle\langle\psi| \tag{1.5}$$

In the present context, the decisive point is now that *a true mixture [being characterized by a sum of terms like (1.5)] can never become a pure*

state. This theorem plays an important role in measurement theory and has been verified repeatedly (e.g., Ghirardi *et al.*, 1976a,b). However, the proof essentially is based upon the Hermiticity of the Hamiltonian $\hat{H}$ and therefore the theorem would not apply within a non-Hermitian approach to quantum theory. Thus the question to be dealt with in the present paper arises whether such a *non-Hermitian* approach to quantum theory exists for which the transitions from mixtures into pure states and vice versa would not be excluded from the very beginning. But on the other hand, would such a non-Hermitian approach not invalidate the basic conservation laws of quantum theory?

As shown in Mattes and Sorg (1994), Ochs and Sorg (n.d.-a), and Sorg (n.d.), the logical link between Hermiticity and conservation laws is not so stringent as could be supposed by the preceding sketch of Schrödinger's nonrelativistic quantum theory. Indeed, insisting on Schrödinger's idea of constructing quantum dynamics, one can easily write down a "*relativistic Schrödinger equation*" (RSE) with a non-Hermitian Hamiltonian $\mathcal{H}_\mu$ [see equation (3.8) below] and nevertheless the desired conservation laws do apply! Though the physical interpretation of the RSE is rather individualistic than statistical, it formally resembles very much its nonrelativistic counterpart (1.1). Especially, the relativistic analogue of the density matrix $\hat{\rho}$ is the "*intensity matrix*" $\mathcal{J}$, which then obeys an equation of motion [see (3.1) below] which may be considered as the relativistic analogue of the evolution equation for the density matrix $\hat{\rho}$, (1.3). But the conclusion (1.4) to be drawn from (1.3) does not hold, on account of the non-Hermiticity inherent in the relativistic approach. Thus the question of *transitions from mixtures into pure states* becomes nontrivial and requires detailed analysis. Unfortunately, we are not able to present the most general answer to the problem, but we can find a *negative* outcome for some special situation ($\rightsquigarrow$ no such transition possible). However, our special result strongly suggests its own generalization, i.e., we expect that within RST the mixtures cannot be dynamically connected to the pure states in the quite general case.

But in contrast to the analogous situation in ordinary quantum mechanics, the disconnectedness of mixtures and pure states in RST does not prohibit a resolution of the well-known paradoxes of the Einstein–Rosen–Podolski type (Selleri, 1990). The reason is here that in RST the disentanglement of the subsystems of some larger system has nothing to do with the question of mixtures and pure states (as in ordinary quantum mechanics), but rather is logically referred to the dynamical (block) diagonalization of the relevant operators such as, e.g., Hamiltonian $\mathcal{H}_\mu$, intensity matrix $\mathcal{J}$, etc. [for a discussion of composite systems see Mattes and Sorg (1995)]. An example for the actual occurrence of such a disentangling diagonalization process has been presented in Sorg (n.d.) and Ochs and Sorg (n.d.-b).

In what follows we briefly define the specific nature of the physical situation considered and present a survey of the organization and results of the paper.

First, we have to observe here that the admittance of a non-Hermitian Hamiltonian $\mathcal{H}_\mu$ surely is a necessary condition for the existence of a dynamical link between mixtures and pure states, but it may not be sufficient. Therefore we should additionally consider also two other important types of effects, *nonlinearity* and *dissipation*, which eventually may become decisive for the transmutation of mixtures and pure states. Concerning the dissipative effects, we will work over an expanding universe. This means that we first review the general form of RST over an arbitrary pseudo-Riemannian spacetime (Sections 2 and 3). Furthermore, at the end of the paper (Section 8) we finally specialize the geometry into a Robertson–Walker universe and additionally consider homogeneous and isotropic field configurations. Here it becomes immediately obvious that the *expansion* of the universe acts as a kind of *friction force* with the Hubble expansion rate H as the damping "constant." However, this procedure will then clearly demonstrate that dissipation is completely unable to produce the transition into pure states. Rather, the suppression of the pure states is due to the Hermitian part of the Hamiltonian (i.e., "kinetic fields"). These kinetic fields equip the subspace of pure states (geometrically represented by the "Fierz cone") with an infinitely strong potential of repulsive nature. As a consequence, the field configuration is permanently kept from becoming a pure state. By analogy to Newtonian point particle mechanics, it is concretely recognized that dissipation cannot help the field configuration climbing up the kinetic potential wall erected over the Fierz cone, which itself stands for the pure states (Fig. 1). On the other hand, it is just the repulsive nature of that kinetic *potential* U_K which makes the pure states unstable, i.e., a small perturbation of a pure state will cause its decay into a mixture.

Next, concerning the nonlinearities, it should be remarked that the question of nonlinear generalizations of conventional quantum mechanics is of continued interest (Weinberg, 1989, and references cited therein). In the context of our special situation of an expanding RW universe, we may think of some nonlinear *Higgs potential* which usually is made responsible for the primeval inflation of the universe (Ochs and Sorg, n.d.-a; Abbott and Pi, 1986; Kolb and Turner, 1990). As is well known, the point with such a Higgs potential is the spontaneous symmetry breakdown, i.e., the field configuration adopts its energetic minimum for some nonzero value of the corresponding field variable. But for the present discussion of the competition between mixtures and pure states, it is instructive also to add another type of nonlinear potential ("*Fierz potential*") which equips the mixtures with an additional energy content relative to the pure states (Section 4). The motivation here is

that the *admittance* of non-Hermiticity of the Hamiltonian $\mathcal{H}_\mu$ would possibly not imply with necessity the transition into a pure state, which, however, might then be enforced by equipping the mixtures with an additional energy. As a consequence the field configuration should now become a pure state whenever it tends to adopt minimal energy. But in any case, the inclusion of nonlinearities of both the Higgs and Fierz type requires the discussion of the most general type of nonlinearity being compatible with the formal structure of RST. This investigation is carried through in Sections 5–7, where the question of conservation laws and energy-momentum density in the presence of nonlinearities is discussed in detail. Here it is found that the constant mass operator $\mathcal{K}$ of the linear theory [$\rightsquigarrow \mathcal{K}_{\text{lin}} = (Mc/\hbar)\cdot\mathbf{1}$] may be generalized into a nonlinear (matrix) function of the intensity matrix $\mathcal{J}$ (the relativistic analogue of the density matrix $\hat{\rho}$).

However, the outcome for the nonlinearities is again negative, similar to the case of the dissipative effects mentioned above (Section 8). The reason is again the same as for dissipation: as long as the nonlinear potentials of Higgs and Fierz type are regular everywhere in space-time, they are dominated by the infinite kinetic potential U_K over the Fierz cone (i.e., the pure states). Thus the repulsive character of that kinetic potential will always dominate over the nonlinear Higgs and Fierz potentials, irrespectively of whether they are attractive or repulsive; and consequently the mixtures are safely separated from the pure states by an infinitely high potential barrier. This barrier is an intrinsic feature of RST, independent of any choice of linear or nonlinear potential, and therefore it is expected that the present impossibility of transitions from mixtures into pure states is a rather general result.

2. INTENSITY MATRIX

In RST, quantum matter is described by the "*intensity matrix*" $\mathcal{J}$, a Hermitian $N \times N$ matrix, $\mathcal{J} = \overline{\mathcal{J}}$, where the dimension N of the typical fiber of the bundle reflects the "degree of complexity" for the physical system (i.e., roughly the particle number). This matrix $\mathcal{J}$ can be thought of as a collection of the "*intrinsic densities*" $\Delta_a(x)$ of the physical system, i.e.,

$$\Delta_a(x) = \text{tr}(\mathcal{J}(x)\cdot\delta_a(x)) \qquad (a = 1, 2, \ldots, N^2) \tag{2.1}$$

Here δ_a ($= \overline{\delta}_a$) constitute a complete set of Hermitian operators over the N-dimensional fiber space. As an example, consider the fiber dimension $N = 2$ and write down the intensity matrix $\mathcal{J}$ as

$$\mathcal{J}(2) = \tfrac{1}{2}(\rho\cdot\mathbf{1} - s_j\sigma^j) \tag{2.2}$$

where σ^j are the usual Pauli matrices

$$\sigma^j\cdot\sigma^k = -g^{jk}\cdot\mathbf{1} + i\epsilon_l^{jk}\sigma^l \qquad \{g^{jk} = \text{diag}(-1, -1, -1)\} \tag{2.3}$$

The *intrinsic densities* ρ, s^j $(= g^{jk}s_k)$ are given by

$$\rho = \mathrm{tr}(\mathcal{J}(2)\cdot\mathbf{1}) \tag{2.4a}$$

$$s^j = \mathrm{tr}(\mathcal{J}(2)\cdot\sigma^j) \tag{2.4b}$$

where the scalar density ρ, (2.4a), as the trace of the intensity matrix $\mathcal{J}$, is gauge invariant and therefore counts also as an observable ($\rightsquigarrow$ "*extrinsic densities*").

An important subclass of intensity matrices is defined through the "*Fierz identity*" (Sorg, n.d.; Fierz, 1937; Crawford, 1985)

$$\mathcal{J}^2 = \rho\mathcal{J} \qquad (\rho := \mathrm{tr}\,\mathcal{J}) \tag{2.5}$$

Obviously this subclass is characterized by the fact that the intensity matrix $\mathcal{J}$ essentially is a one-dimensional projector, i.e.,

$$\mathcal{J}(x) = \psi(x) \otimes \bar{\psi}(x) \qquad (\rightsquigarrow \rho \equiv \bar{\psi}\cdot\psi) \tag{2.6}$$

with $\psi(x)$ being some section of the associated N-dimensional complex vector bundle over space-time. Observe also that the projector property (2.6) admits the existence of some orthogonal operator $\mathcal{G}$ ["*convertor*" (Sorg, n.d.)], so that

$$\mathcal{G}\cdot\mathcal{J} = \mathcal{J}\cdot\mathcal{G} \equiv 0 \tag{2.7a}$$

or

$$\mathcal{G}\cdot\psi \equiv 0 \tag{2.7b}$$

On the other hand, it follows trivially from equation (2.6) that the "*wave function*" $\psi(x)$ is an eigenvector of $\mathcal{J}$,

$$\mathcal{J}\cdot\psi = \rho\psi \tag{2.8}$$

and this immediately transfers to an arbitrary matrix function $\mathcal{X}$ of the intensity matrix $\mathcal{J}$, i.e.,

$$\mathcal{X}(\mathcal{J})\cdot\psi = X(\rho)\psi \tag{2.9}$$

where the real function $X(\rho)$ denotes the same functional dependence as its matrix counterpart $\mathcal{X}(\mathcal{J})$.

Clearly, when the intensity matrix $\mathcal{J}$ is nothing else than the collection of intrinsic densities and furthermore obeys the special relationship (2.5), there must exist certain constraints among those densities just as a consequence of that Fierz identity. For instance, return for the moment to the two-dimensional case (2.2) and find from the Fierz identity (2.5)

$$0 = \rho^2 + s^j s_j \equiv \rho^2 - s^2 \tag{2.10}$$

Thus, if the extrinsic densities ρ, s would be accessible to experiment (or at

least to observation), one could test whether the physical system admits a wave function ($\psi(x)$ ("*pure state*") or whether it is a "*mixture*" ($\rightsquigarrow$ Fierz identity not satisfied). As we shall see, the RST predicts that the pure states constitute a somewhat singular and unstable subset of all dynamically possible states, i.e., physical systems should always occur as mixtures (possibly close to pure states). However, the question of the stability of the pure states is decided by the equations of motion, which therefore have to be discussed now.

3. EQUATIONS OF MOTION

The problem of how matter behaves under the action of some "field strength" $\mathcal{F}_{\mu\nu}$ is solved by the field equation for the intensity matrix $\mathcal{I}$:

$$\mathcal{D}_\mu \mathcal{I} = \frac{i}{\hbar c} [\mathcal{I} \cdot \overline{\mathcal{H}}_\mu - \mathcal{H}_\mu \cdot \mathcal{I}] \tag{3.1}$$

Here, $\mathcal{D}_\mu$ denotes the gauge-covariant derivative

$$\mathcal{D}_\mu \mathcal{I} := \partial_\mu \mathcal{I} + [\mathcal{A}_\mu, \mathcal{I}] \tag{3.2}$$

with the "*gauge potential*" $\mathcal{A}_\mu$ generating the field strength $\mathcal{F}_{\mu\nu}$ as usual,

$$\mathcal{F}_{\mu\nu} = \nabla_\mu \mathcal{A}_\nu - \nabla_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{3.3}$$

Moreover the (non-Hermitian) "*Hamiltonian*" $\mathcal{H}_\mu$ is itself a dynamical object in RST (contrary to Schrödinger's Hamiltonian operator $\hat{H}$) and the corresponding field equations consist of the "*integrability condition*"

$$\mathcal{D}_\mu \mathcal{H}_\nu - \mathcal{D}_\nu \mathcal{H}_\mu + \frac{i}{\hbar c} [\mathcal{H}_\mu, \mathcal{H}_\nu] = i\hbar c \mathcal{F}_{\mu\nu} \tag{3.4}$$

and the "*conservation equation*"

$$\mathcal{D}^\mu \mathcal{H}_\mu - \frac{i}{\hbar c} \mathcal{H}^\mu \cdot \mathcal{H}_\mu = -i\hbar c(\mathcal{K} + i\mathcal{G}) \tag{3.5}$$

Obviously, the Hamiltonian dynamics introduces two new (Hermitian) objects, namely the "*mass operator*" $\mathcal{K}$ $(= \overline{\mathcal{K}})$ and the "*convertor*" $\mathcal{G}$ $(= \overline{\mathcal{G}})$ and consequently we have to specify their equations of motion in order to close the whole dynamical system.

For that purpose, consider first the convertor $\mathcal{G}$. For a closed matter system (whose rest mass M remains invariant) the field equation for $\mathcal{G}$ is chosen as (Sorg, n.d.)

$$\mathcal{D}_\mu \mathcal{G} = \frac{i}{\hbar c} (\mathcal{G} \cdot \mathcal{H}_\mu - \overline{\mathcal{H}}_\mu \cdot \mathcal{G}) \tag{3.6}$$

so that the algebraic constraint (2.7a) is actually obeyed over all space-time

if it holds at one event. Evidently, it is just that algebraic constraint (2.7a) which, albeit somewhat indirectly, additionally couples the intensity matrix $\mathcal{I}$ to the Hamiltonian $\mathcal{H}_\mu$, namely via the convertor $\mathcal{G}$ in the conservation equation (3.5). But whenever the mass operator $\mathcal{X}$ is thought to be independent of the densities Δ_a, (2.1), i.e., for the linear theory,

$$\mathcal{X} \to \mathcal{X}_{\text{lin}} = \left(\frac{Mc}{\hbar}\right)^2 \cdot \mathbf{1} \tag{3.7}$$

the coupled Hamiltonian and convertor dynamics (3.4)–(3.6) constitutes a *closed* operator system which is completely independent of the intrinsic densities Δ_a or the wavefunction ψ. As a consequence one can solve the problem of motion for matter in two steps: (i) first solve the equations of motion for the operators $\mathcal{H}_\mu$ and $\mathcal{G}$, and (ii) then solve the equation of motion for the intensity matrix $\mathcal{I}$, (3.1), with the algebraic constraint (2.7a), or (2.7b), as an initial condition. For the special case (2.6) where a wave function ψ exists, the corresponding equation of motion for ψ is the "*relativistic Schrödinger equation*" (RSE)

$$i\hbar c \mathcal{D}_\mu \psi = \mathcal{H}_\mu \psi \tag{3.8}$$

which can easily be deduced from the intensity dynamics (3.1) (Sorg, n.d.). The crucial point is now that, despite the fact that Hamiltonian dynamics (3.4)–(3.5) is a nonlinear system, one can have a linear theory by dealing exclusively with the wave function ψ. In order to see this more clearly, differentiate once more the RSE (3.8) and find by means of the conservation equation (3.5) and the constant mass operator $\mathcal{X}_{\text{lin}}$, (3.7),

$$\mathcal{D}^\mu \mathcal{D}_\mu \psi + \left(\frac{Mc}{\hbar}\right)^2 \psi = 0 \tag{3.9}$$

which is nothing else than the *linear* Klein–Gordon equation (KGE). Obviously, there is an intimate correspondence between the linearity of the wave equation and the closedness of the operator dynamics.

However, modern gauge field theories [Salam–Weinberg model (Quigg, 1983), GUTs (Huang, 1982), etc.] convincingly demonstrate that nonlinear wave equations play a relevant role in particle physics and cosmology (Abbott and Pi, 1986; Kolb and Turner, 1990). Therefore it becomes adequate to conceive the mass operator $\mathcal{X}$ containing also the nonlinearities due to some Higgs potential V_H, i.e., we may put

$$\mathcal{X} \to \mathcal{X}_H = \mathcal{X}_H(\mathcal{I}) \tag{3.10}$$

Here, the Higgs mass operator $\mathcal{X}_H(\mathcal{I})$ is some matrix function of the intensity matrix $\mathcal{I}$, e.g., a polynomial

$$\mathscr{K}_{\mathrm{H}}(\mathcal{S}) = 2\left(\frac{Mc}{\hbar}\right)^2 (2a_{\mathrm{H}}^3 \mathcal{S} - \mathbf{1}) \tag{3.11}$$

where a_{H} is a constant length parameter. Such a nonlinear choice for $\mathscr{K}$ generalizes the ordinary KGE (3.9) into the nonlinear Klein–Gordon–Higgs equation (KGHE)

$$\mathscr{D}^{\mu}\mathscr{D}_{\mu}\psi + X(\rho)\psi = 0 \tag{3.12}$$

where the ordinary real function $X_{\mathrm{H}}(\rho)$ is deduced from the Higgs choice (3.11) as

$$X_{\mathrm{H}}(\rho) = 2\left(\frac{Mc}{\hbar}\right)^2 (2a_{\mathrm{H}}^3 \rho - 1) \tag{3.13}$$

[see the arguments concerning equations (2.8)–(2.9)]. Observe, however, that in RST one is *not* forced to apply the Fierz identity. For instance, for the fiber dimension $N = 2$ we can admit a general mixture where the densities ρ and s^j of (2.4a) and (2.4b) are not subject to the Fierz condition (2.10). Therefore they enter the Hamiltonian dynamics (3.5) in an unrestricted way via the intensity matrix $\mathcal{S}(2)$, (2.2), and the Higgs mass operator $\mathscr{H}_{\mathrm{H}}(\mathcal{S})$, (3.11). However, this implies that the operator dynamics (3.4)–(3.6) is no longer closed, but becomes coupled to the intrinsic densities Δ_a. As a consequence of this feedback between operators and densities the two-step procedure of the linear theory no longer applies and both quantities must be determined simultaneously.

4. FIERZ POTENTIAL

Originally, the introduction of a Higgs potential was motivated by the idea that the energy of the corresponding field configuration be minimized by some *nonzero* value of the field variable. Through this mechanism, the highly welcome effect of spontaneous symmetry breaking could be achieved together with all its important implications (Quigg, 1983; Huang, 1982). In the present context, one can transfer the Higgs idea to the case of pure states within the larger set of mixture configurations. More concretely, the idea is that the configuration of lowest energy should be a pure state and not a mixture. But this would imply that the mixtures are equipped with an additional energy content in comparison to the pure states, i.e., there exists a "*Fierz potential*" V_{F} which is always positive for mixtures, but vanishes for the pure states.

In order to construct such a potential, we first define the deviation operator $\mathscr{D}_{\mathrm{F}}$ ("*deviator*") through

$$\mathscr{D}_{\mathrm{F}} := \rho\mathcal{S} - \mathcal{S}^2 \tag{4.1}$$

The Fierz "*deviation*" Δ_F as a measure for the invalidation of the Fierz identity (2.5) is then defined in a similar manner through

$$\Delta_F = \mathrm{tr}(\mathfrak{D}_F) \tag{4.2}$$

To give an example, consider again the case $N = 2$ and find for the Fierz deviation $\Delta_F^{(2)}$ in two dimensions

$$\Delta_F^{(2)} := 2\,\mathrm{tr}(\mathfrak{D}_F(2)) = \rho^2 - s^2 \tag{4.3}$$

which naturally vanishes for pure states; cf. (2.10).

Turning now to the task of constructing some nonnegative potential V_F as a function of the deviation Δ_F, one may take

$$V_F = (Mc^2)a_F^9\,\mathrm{tr}\{(\mathfrak{D}_F)^2\} \tag{4.4}$$

where a_F is a constant length parameter. Since the square of a Hermitian operator ($\mathfrak{D}_F = \overline{\mathfrak{D}_F}$) safely is a positive operator, the Fierz potential V_F of (4.4) must necessarily be positive and can vanish only for $\mathfrak{D}_F = 0$, i.e., for pure states. For instance, the Fierz potential in two dimensions becomes

$$\begin{aligned} V_F^{(2)}(\rho, s) &= Mc^2a_F^9\,\mathrm{tr}\{\rho^2(\mathfrak{H}(2))^2 - 2\rho(\mathfrak{H}(2))^3 + (\mathfrak{H}(2))^4\} \\ &= Mc^2\left(\frac{a_F^3}{2}\right)^3(\rho^2 - s^2)^2 \\ &\equiv Mc^2\left(\frac{a_F^3}{2}\right)^3(\Delta_F^{(2)})^2 \end{aligned} \tag{4.5}$$

For pure states, we have $\rho = \pm s$ ($\Rightarrow \Delta_F^{(2)} = 0$) and hence $V_F^{(2)}$ vanishes, as expected.

Clearly the presence of any nonlinear potential term in the theory, such as V_H or V_F, is accompanied by the corresponding nonlinear mass operator $\mathfrak{X}_H$ and $\mathfrak{X}_F$, respectively. So the total mass operator $\mathfrak{X}$ will be found as the sum of both contributions

$$\mathfrak{X}(\mathfrak{H}) = \mathfrak{X}_H(\mathfrak{H}) + \mathfrak{X}_F(\mathfrak{H}) \tag{4.6}$$

due to the total potential

$$V(\rho, s) = V_H(\rho, s) + V_F(\rho, s) \tag{4.7}$$

The Higgs mass operator $\mathfrak{X}_H(\mathfrak{H})$ has already been specified by equation (3.11) and the Fierz potential V_F by (4.4), but how does one associate the Higgs potential $V_H(\rho, s)$ to the mass operator $\mathfrak{X}_H(\mathfrak{H})$? Or equivalently, how does one find the Fierz mass operator $\mathfrak{X}_F(\mathfrak{H})$ from the Fierz potential V_F? The answer to these questions comes from a closer inspection of the energy-

momentum balance: i.e., we have to consider now the conservation laws for the nonlinear theory.

5. CONTINUITY EQUATIONS

The simplest conservation law refers to the invariance of certain scalar quantities such as electric charge or rest mass (Sorg, n.d.). Associated to such a scalar invariant is some current density j_μ which has to obey the well-known continuity equation

$$\nabla^\mu j_\mu \equiv 0 \tag{5.1}$$

In the true spirit of RST, such extrinsic densities as the current j_μ are gauge-invariant local observables generated by the associated operators, i.e., in the present case of the "*velocity operator*" v_μ $(= \bar{v}_\mu)$:

$$j_\mu = \mathrm{tr}(\mathcal{J} \cdot v_\mu) \tag{5.2a}$$

or, if a wave function $\psi(x)$ exists,

$$j_\mu = \bar{\psi} \cdot v_\mu \cdot \psi \tag{5.2b}$$

The continuity requirement (5.1) is readily transcribed to the velocity operator v_μ and then reads by use of the equation of motion for $\mathcal{J}$, (3.1), or for ψ, (3.8),

$$\mathcal{D}^\mu v_\mu + \frac{i}{\hbar c}[\overline{\mathcal{H}}^\mu \cdot v_\mu - v_\mu \cdot \mathcal{H}^\mu] = \mathcal{G}' \tag{5.3}$$

where the operator $\mathcal{G}'$ annihilates again the intensity matrix

$$\mathcal{G}' \cdot \mathcal{J} \equiv 0 \tag{5.4a}$$

or the wave function ψ, respectively,

$$\mathcal{G}' \cdot \psi \equiv 0 \tag{5.4b}$$

quite similarly as for $\mathcal{G}$, (2.7a)–(2.7b). Consequently $\mathcal{G}'$ should obey again the equation of motion

$$\mathcal{D}_\mu \mathcal{G}' = \frac{i}{\hbar c}(\mathcal{G}' \cdot \mathcal{H}_\mu - \overline{\mathcal{H}}_\mu \cdot \mathcal{G}') \tag{5.5}$$

applying to the Hermitian operators [cf. (3.6)], in order to guarantee the validity of the algebraic constraints (5.4a)–(5.4b) over the whole space-time.

One will be interested in keeping the number of dynamical objects as small as possible and therefore one will try to identify the new velocity operator v_μ with some object already present in the theory. To this end, the Hamiltonian $\mathcal{H}_\mu$ is split up into its (anti-) Hermitian parts $\mathcal{K}_\mu$, $\mathcal{L}_\mu$

$$\mathcal{H}_\mu = \hbar c(\mathcal{K}_\mu + i\mathcal{L}_\mu) \tag{5.6}$$

and then the continuity requirement (5.3) for the velocity operator v_μ reads

$$\mathcal{D}^\mu v_\mu + i[\mathcal{H}^\mu, v_\mu] + \{\mathcal{L}^\mu, v_\mu\} = \mathcal{G}' \tag{5.7}$$

On the other hand, the Hamiltonian splitting (5.6) generates two coupled equations of motion from the single equation (3.5), namely one for the "*kinetic field*" $\mathcal{H}_\mu$

$$\mathcal{D}^\mu \mathcal{H}_\mu + \{\mathcal{L}^\mu, \mathcal{H}_\mu\} = \mathcal{G} \tag{5.8}$$

and the other for the "*localization field*" $\mathcal{L}_\mu$

$$\mathcal{D}^\mu \mathcal{L}_\mu + \mathcal{L}^\mu \mathcal{L}_\mu - \mathcal{H}^\mu \mathcal{H}_\mu = -\mathcal{K} \tag{5.9}$$

But this result provides us now with a possible solution for the velocity operator v_μ: comparing its field equation (5.7) with that for the kinetic field $\mathcal{H}_\mu$, (5.8), suggests that we take

$$v_\mu = \frac{\hbar}{Mc} \mathcal{H}_\mu \tag{5.10}$$

with

$$\mathcal{G}' = \frac{\hbar}{Mc} \mathcal{G} \tag{5.11}$$

This result for the velocity operator v_μ exhibits some interesting features especially concerning the conservation of rest mass. First define the "*four-momentum operator*" $\mathcal{P}_\mu$ through

$$\mathcal{P}_\mu := Mcv_\mu = \hbar \mathcal{H}_\mu \tag{5.12}$$

and then split up the integrability condition (3.4) into its (anti-) Hermitian parts yielding for $\mathcal{P}_\mu$

$$\mathcal{D}_\mu \mathcal{P}_\nu - \mathcal{D}_\nu \mathcal{P}_\mu + \frac{i}{\hbar} [\mathcal{P}_\mu, \mathcal{P}_\nu] = i\hbar \mathcal{F}_{\mu\nu} \tag{5.13}$$

and for the localization field $\mathcal{L}_\mu$

$$\mathcal{D}_\mu \mathcal{L}_\nu - \mathcal{D}_\nu \mathcal{L}_\mu + i[\mathcal{L}_\mu, \mathcal{H}_\nu] + i[\mathcal{H}_\mu, \mathcal{L}_\nu] = 0 \tag{5.14}$$

Obviously the Hermitian part (5.13) establishes a link between the derivatives of four-momentum $\mathcal{P}_\mu$ and field strength $\mathcal{F}_{\mu\nu}$ and thus may be considered as the relativistic counterpart of Ehrenfest's theorem for the nonrelativistic Schrödinger theory. Indeed, it has been shown that the well-known equation of motion for a classical point particle in an electromagnetic field $\mathcal{F}_{\mu\nu}$

$$\frac{dp_\nu}{d\tau} = \frac{e}{Mc} F_{\mu\nu} p^\mu \tag{5.15}$$

where τ is the proper time along the particle world-line, follows immediately from the relativistic Ehrenfest relation (5.13) just in the classical limit ($\hbar \to 0$) (Mattes and Sorg, 1995). But, clearly, the classical Lorentz force on the right of (5.15) does not change the rest mass M of the system (see Section 7). On the other hand, the full RST does admit the change of rest mass M (particle creation/annihilation), which may be readily seen by defining the mass current ${}^{(C)}j_\mu$ through

$${}^{(C)}j_\mu := \mathrm{tr}(\mathcal{J} \cdot \mathcal{P}_\mu) = Mc \cdot j_\mu \tag{5.16}$$

and computing its source (sink) as

$$\nabla^\mu \, {}^{(C)}j_\mu = \hbar \, \mathrm{tr}(\mathcal{J} \cdot \mathcal{G}) \tag{5.17}$$

Thus there is no conservation of rest mass M in RST if the algebraic constraint (2.7a)–(2.7b) does not hold. In this sense it becomes plausible that the convertor $\mathcal{G}$ will govern the "*conversion*" of rest mass into gauge field energy and vice versa (particle creation/annihilation), which, however, cannot be described by the "classical approximation" (5.15).

Let us stop here for a moment in order to make the present results concrete by means of our previous example of fiber dimension $N = 2$. The kinetic and localization fields $\mathcal{H}_\mu$, $\mathcal{L}_\mu$ of (5.6) may be decomposed here with respect to the $N^2 = 4$ Hermitian operators $\{\mathbf{1}, \sigma^j\}$ as

$$\mathcal{H}_\mu^{(2)} = K_\mu \cdot \mathbf{1} + K_{j\mu}\sigma^j \tag{5.18a}$$

$$\mathcal{L}_\mu^{(2)} = L_\mu \cdot \mathbf{1} + L_{j\mu}\sigma^j \tag{5.18b}$$

and similarly the convertor $\mathcal{G}$ adopts the special form of a projector ($\mathcal{G}^2 = G\mathcal{G}$, $G = \mathrm{tr}\,\mathcal{G}$):

$$\mathcal{G}(2) = \tfrac{1}{2}G(\mathbf{1} + \hat{s}_j\sigma^j); \qquad s_j = s\hat{s}_j, \quad \hat{s}^j\hat{s}_j = -1 \tag{5.19}$$

just on account of the algebraic constraint (2.7a)–(2.7b). This constraint is also responsible for the fact that the intensity matrix $\mathcal{J}(2)$, (2.2), must necessary be a projector

$$\mathcal{J}(2) = \tfrac{1}{2}\rho(\mathbf{1} - \hat{s}_j\sigma^j) \tag{5.20}$$

and thus both matrices $\mathcal{G}$ and $\mathcal{J}$ obey the Fierz identity (2.5). With these arrangements, the mass current density ${}^{(C)}j_\mu$ in (5.16) reads

$${}^{(C)}j_\mu = \hbar(\rho K_\mu + s^j K_{j\mu}) \tag{5.21}$$

and its source is found from this by means of the Hamiltonian dynamics (5.8) and of the density dynamics as

$$\nabla^{\mu\,(C)}j_{\mu} = \tfrac{1}{2}G(\rho - s) \tag{5.22}$$

From here we again recognize that the presence of a nontrivial convertor $\mathcal{G}(2)$ (i.e., $G \neq 0$) implies the Fierz identity (i.e., $\rho \equiv s$). Thus for our subsequent discussion of the *mixtures* $\rho \neq s$ we must put the convertor $\mathcal{G}(2)$ identically to zero!

6. THE GAUGE FIELD

Evidently the preceding continuity equations are completely indifferent to the presence of nonlinearities, i.e., the central equation (5.1) is valid irrespective of whether we want to apply a nonlinear mass term $\mathcal{X}(\mathcal{J})$, (3.11), or prefer to have a linear theory $\mathcal{X}_{\text{lin}}$, (3.7). We want to demonstrate the extent to which this indifference will persist when turning now to the coupling of the gauge and matter fields. Considering the general belief that the sources $\mathcal{J}_{\nu}$ of the field strength $\mathcal{F}_{\mu\nu}$, (3.3), should be composed of the matter field $\psi(x)$, or more generally, intensity matrix $\mathcal{J}(x)$, one writes down the Yang–Mills equations for $\mathcal{F}_{\mu\nu}$ as usual[2]

$$\mathcal{D}^{\mu}\mathcal{F}_{\mu\nu} = 4\pi\alpha\mathcal{J}_{\nu} \tag{6.1}$$

where $\alpha = g^2/\hbar c$ is the fine structure constant for the corresponding type of interaction. The problem is now to construct the "gauge current" $\mathcal{J}_{\nu}$ in terms of the intensity matrix $\mathcal{J}$.

For that purpose one should first consider the general constraints to be obeyed by any gauge current $\mathcal{J}_{\nu}$. The most important constraint comes from the bundle identity for the curvature $\mathcal{F}_{\mu\nu}$:

$$[\mathcal{D}_{\lambda}\mathcal{D}_{\sigma} - \mathcal{D}_{\sigma}\mathcal{D}_{\lambda}]\mathcal{F}_{\mu\nu} \equiv [\mathcal{F}_{\lambda\sigma}, \mathcal{F}_{\mu\nu}] - R^{\rho}_{\mu\lambda\sigma}\mathcal{F}_{\rho\nu} - R^{\rho}_{\nu\lambda\sigma}\mathcal{F}_{\mu\rho} \tag{6.2}$$

where $R^{\rho}_{\mu\lambda\sigma}$ denotes the Riemannian curvature tensor of the underlying space-time. The double tensor contraction of this identity yields

$$\mathcal{D}^{\nu}\mathcal{D}^{\mu}\mathcal{F}_{\mu\nu} \equiv 0 \tag{6.3}$$

and this immediately implies the generalized continuity equation for the gauge current $\mathcal{J}_{\mu}$,

$$\mathcal{D}^{\mu}\mathcal{J}_{\mu} \equiv 0 \tag{6.4}$$

[2] Since the physical concepts of "field strength" and "current density" have been defined independently of the geometric notion of "bundle curvature," one needs some dimensional constant in order to fit together the physical and geometric quantities when they appear in the same equation; see, e.g., equations (6.15a)–(6.15c).

provided we want to insist on the Yang–Mills equations (6.1). Thus, the problem is now to construct the gauge current $\mathcal{J}_\mu$ in terms of the intensity matrix $\mathcal{J}$ in such a way that the identity (6.4) is automatically satisfied. In order to find the corresponding ansatz for $\mathcal{J}_\mu$, both the curvature $\mathcal{F}_{\mu\nu}$ and the gauge current $\mathcal{J}_\mu$ are decomposed with respect to the (anti-Hermitian) generators τ^a of the holonomy group (for the connection $\mathcal{A}_\mu$) as

$$\mathcal{F}_{\mu\nu} = F_{a\mu\nu}\tau^a \tag{6.5a}$$

$$\mathcal{J}_\mu = j_{a\mu}\tau^a \qquad (\mathcal{A}_\mu = A_{a\mu}\tau^a) \tag{6.5b}$$

Denoting the structure constants of the holonomy algebra by C_c^{ab}, we have

$$[\tau^a, \tau^b] = C_c^{ab}\tau^c \tag{6.6a}$$

$$\mathcal{D}_\mu\tau^a = 0 \tag{6.6b}$$

The Yang–Mills equation (6.1) establishes now the corresponding link for the curvature and current coefficients $F_{a\mu\nu}$, $j_{a\mu}$ as

$$D^\mu F_{a\mu\nu} = 4\pi\alpha j_{a\nu} \tag{6.7}$$

and the gauge continuity equation (6.4) reads

$$D^\mu j_{a\mu} = 0 \qquad (D_\mu j_{a\nu} := \nabla_\mu j_{a\nu} + C_a^{bc}A_{b\mu}j_{c\nu}) \tag{6.8}$$

But now the construction of the desired gauge current $j_{a\mu}$ is easily achieved by reference to the corresponding gauge velocity operator $v_{a\mu}$

$$j_{a\mu} = \mathrm{tr}(\mathcal{J}\cdot v_{a\mu}) \tag{6.9}$$

to be subjected to the corresponding source equation

$$\mathrm{tr}\left\{\mathcal{J}\cdot\left(\mathcal{D}^\mu v_{a\mu} + \frac{i}{\hbar c}[\overline{\mathcal{H}}^\mu\cdot v_{a\mu} - v_{a\mu}\cdot\mathcal{H}^\mu]\right)\right\} \equiv 0 \tag{6.10}$$

This requirement is immediately deduced again from the continuity requirement for $j_{a\mu}$, (6.8), by means of the field equation for the intensity matrix $\mathcal{J}$, (3.1). Since the arguments run here quite analogously to the previous case of the original continuity equation (5.1), we can apply the same reasoning again and obtain from the present requirement (6.10)

$$\mathcal{D}^\mu v_{a\mu} + \frac{i}{\hbar c}[\overline{\mathcal{H}}^\mu\cdot v_{a\mu} - v_{a\mu}\cdot\mathcal{H}^\mu] = \frac{\hbar}{2Mc}([\tau_a, \mathcal{K}] + i\{\tau_a, \mathcal{G}\}) \tag{6.11}$$

provided the gauge velocity operators $v_{a\mu}$ are introduced as follows:

$$v_{a\mu} = \frac{i}{2Mc^2}\{\overline{\mathcal{H}}_\mu\cdot\tau_a + \tau_a\cdot\mathcal{H}_\mu\} \tag{6.12}$$

Thus the original continuity requirement (6.8) becomes

$$0 \overset{!}{=} D^{\mu} j_{a\mu} = \frac{\hbar}{2Mc} \{\mathrm{tr}([\mathscr{X}, \mathscr{J}] \cdot \tau_a) + i\, \mathrm{tr}(\{\mathscr{G}, \mathscr{J}\} \cdot \tau_a)\} \tag{6.13}$$

and this requirement can actually be satisfied for the following reasons:

(i) The mass operator $\mathscr{X}$ is chosen to be a matrix function of the intensity matrix $\mathscr{J}$ [i.e., $\mathscr{X} = \mathscr{X}(\mathscr{J})$]:

$$[\mathscr{X}(\mathscr{J}), \mathscr{J}] \equiv 0 \tag{6.14}$$

[cf. (3.11)].

(ii) The convertor $\mathscr{G}$ and intensity matrix $\mathscr{J}$ are assumed to annihilate each other; cf. (2.7a)–(2.7b).

Summarizing, we see that we have achieved our aim to construct the gauge currents $j_{a\mu}$ in such a way that they automatically obey the continuity requirement (6.8) and thus can be used as the sources for the field strength $\mathscr{F}_{\mu\nu}$, (6.7). Let us demonstrate the consistency of the present result by considering two simple examples: the electromagnetic and the weak interactions. Concerning electromagnetism, the original velocity operator v_{μ}, (5.10), is nothing else than a special case of the gauge velocity operators $v_{a\mu}$, (6.12); this is readily seen by simply taking the holonomy generator as $\tau_a \to -i \cdot \mathbf{1}$. Since this generator can be assumed to be due to the $\mathscr{U}(1)$ gauge group of ordinary Maxwellian electromagnetism (with fine structure constant $\alpha \to e^2/\hbar c$), we put

$$\mathscr{F}_{\mu\nu} \Rightarrow -\frac{ie}{\hbar c} F_{\mu\nu} \cdot \mathbf{1} \tag{6.15a}$$

$$\mathscr{A}_{\mu} \Rightarrow -\frac{ie}{\hbar c} A_{\mu} \cdot \mathbf{1} \tag{6.15b}$$

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \tag{6.15c}$$

and thus the corresponding current j_{μ}, (5.2a)–(5.2b), adopts its original electromagnetic meaning with the Yang–Mills equation (6.1) reducing to the original Maxwell equation (over flat space-time)

$$\partial^{\mu} F_{\mu\nu} = \frac{4\pi}{c} {}^{(\mathrm{el})}j_{\mu} \tag{6.16}$$

The electromagnetic current ${}^{(\mathrm{el})}j_{\mu}$ is given in terms of the associated convection current j_{μ}, (5.2a)–(5.2b), as

$${}^{(\mathrm{el})}j_{\mu} = ecj_{\mu} = \frac{e}{M} {}^{(\mathrm{C})}j_{\mu} \tag{6.17}$$

Consequently, in Abelian electrodynamics the mass current ${}^{(\mathrm{C})}j_{\mu}$ is propor-

tional to its electromagnetic counterpart ${}^{(el)}j_\mu$. Moreover, the gauge-covariant derivative of the wave function $\psi(x)$ becomes

$$\begin{aligned} \mathcal{D}_\mu\psi &\equiv \partial_\mu\psi + \mathcal{A}_\mu\psi \\ &= \partial_\mu\psi - \frac{ie}{\hbar c} A_\mu\psi \end{aligned} \tag{6.18}$$

which is nothing else than the well-known prescription of "*minimal coupling*" of conventional quantum mechanics

$$\hat{p}_\mu \to \hat{p}_\mu - \frac{e}{c} A_\mu \qquad (\hat{p}_\mu = i\hbar\partial_\mu) \tag{6.19}$$

Similarly, for the weak interactions we use the gauge group $SU(2)$ and therefore we put here (for fiber dimension $N = 2$)

$$\tau^a \Rightarrow \tau^j = -\tfrac{1}{2}\sigma^j, \qquad j = 1, 2, 3 \tag{6.20}$$

which yields for the gauge velocity operators (6.12)

$$v_\mu^j = \frac{\hbar}{4Mc}\{\sigma^j, \mathcal{H}_\mu\} + i\frac{\hbar}{4Mc}[\sigma^j, \mathcal{L}_\mu] \tag{6.21}$$

Strange to say, these gauge velocity operators v_μ^j depend also upon the *localization* field $\mathcal{L}_\mu$, whereas the original velocity operator v_μ, (5.10), is linked only to the *kinetic* field $\mathcal{H}_\mu$. Naturally one would expect that the notion of "*velocity*" is in some sense complementary to the concept of "*localization*"; but the general form (6.12) for the gauge velocity operators $v_{a\mu}$ is mandatory for the validity of the gauge continuity relation (6.8) and therefore we have to accept the dependence of the $v_{a\mu}$ upon $\mathcal{L}_\mu$. More concretely, the decomposition (5.18a)–(5.18b) of $\mathcal{H}_\mu$ and $\mathcal{L}_\mu$ with respect to the Hermitian operator basis $\{\mathbf{1}, \sigma^j\}$ readily yields for the weak velocity operators v_μ^j, (6.21),

$$v_\mu^j = \frac{\hbar}{2Mc}(-K_\mu^j \cdot \mathbf{1} + K_\mu \cdot \sigma^j - \epsilon_k^{jl} L_{l\mu}\sigma^k) \tag{6.22}$$

i.e., the corresponding gauge current j_μ^k, (6.9), becomes

$$j_\mu^k = \frac{\hbar}{2Mc}(s^k \cdot K_\mu - \rho \cdot K_\mu^k - \epsilon_j^{kl} L_{l\mu} s^j) \tag{6.23}$$

As a confidence test for this result, one would like to explicitly compute the left-hand side of the continuity equation (6.8); but for this process one first needs the dynamical equations for the intensity matrix $\mathcal{J}(2)$, (3.1), as

well as for $\mathcal{H}_\mu$, (5.8), and $\mathcal{L}_\mu$ (5.9), in component form. Here, the density dynamics is easily deduced from equation (3.1) as

$$\partial_\mu \rho = 2(\rho L_\mu + s^j L_{j\mu}) \tag{6.24a}$$

$$D_\mu s_j = 2\epsilon_j^{kl} K_{k\mu} s_l - 2(\rho L_{j\mu} - s_j L_\mu) \tag{6.24b}$$

and similarly the Hamiltonian dynamics (5.8)–(5.9) reads in coefficient form for the kinetic field

$$\nabla^\mu K_\mu + 2(L^\mu K_\mu - L^{j\mu} K_{j\mu}) = \tfrac{1}{2} G \tag{6.25a}$$

$$D^\mu K_{j\mu} + 2(L^\mu K_{j\mu} + K^\mu L_{j\mu}) = -\tfrac{1}{2} G_j \tag{6.25b}$$

and for the localization field

$$\nabla^\mu L_\mu + (L^\mu L_\mu - L^{j\mu} L_{j\mu}) - (K^\mu K_\mu - K^{j\mu} K_{j\mu}) = -X \tag{6.26a}$$

$$D^\mu L_{j\mu} + 2(L^\mu L_{j\mu} - K^\mu K_{j\mu}) = -X_j \tag{6.26b}$$

Thus the continuity relation (6.8) actually may be verified by covariantly differentiating the preceding gauge current j^k_μ, (6.23), and using the present derivatives (6.26a)–(6.26b), which first yields

$$D^\mu j^k_\mu = \frac{\hbar}{2Mc} \operatorname{tr}(\mathcal{J} \cdot \mathcal{G} \cdot \sigma^k) \tag{6.27}$$

just in agreement with the general requirement (6.13). But now we remember again the algebraic condition (2.7a) and thus we see that the desired continuity relation (6.8) really holds, quite similarly to the case for the original continuity equation (5.17). Obviously the algebraic relationship (2.7a)–(2.7b) plays a central role also for the conservation laws, now only for the deduction of the KGHE (3.12). This effect will show up even more distinctly when considering subsequently the energy-momentum exchange between the matter and gauge fields.

The energy-momentum density of the weak gauge fields is as usual

$${}^{(\mathrm{F})}T_{\mu\nu} = \frac{\hbar c}{2\pi\alpha} \operatorname{tr}\left(\mathcal{F}_{\mu\lambda} \cdot \mathcal{F}_\nu^\lambda - \frac{1}{4} g_{\mu\nu} \mathcal{F}_{\lambda\sigma} \cdot \mathcal{F}^{\lambda\sigma} \right) \tag{6.28a}$$

or in terms of the curvature coefficients $F_{a\mu\lambda}$, (6.5a),

$${}^{(\mathrm{F})}T_{\mu\nu} = \frac{\hbar c}{4\pi\alpha} \left(F_{a\mu\lambda} F^{a\ \lambda}_{\ \nu} - \frac{1}{4} g_{\mu\nu} F_{a\sigma\lambda} F^{a\sigma\lambda} \right) \tag{6.28b}$$

[putting $\tau^a\tau^b + \tau^b\tau^a = (1/N)g^{ab}\cdot\mathbf{1}$]. Consequently, the source of this density ${}^{(\mathrm{F})}T_{\mu\nu}$ becomes

$$\nabla^\mu\,{}^{(\mathrm{F})}T_{\mu\nu} = \hbar c\,\mathrm{tr}\{\mathcal{F}_{\nu\lambda}\cdot\mathcal{J}^\lambda + \mathcal{J}^\lambda\cdot\mathcal{F}_{\nu\lambda}\} \tag{6.29}$$
$$= -\hbar c F_{a\lambda\nu}j^{a\lambda} := -{}^{(\mathrm{L})}f_\nu$$

where the Yang–Mills equations (6.1) have been used together with the well-known Bianchi identity

$$\mathcal{D}_\lambda\mathcal{F}_{\mu\nu} + \mathcal{D}_\mu\mathcal{F}_{\nu\lambda} + \mathcal{D}_\nu\mathcal{F}_{\lambda\mu} \equiv 0 \tag{6.30}$$

The significance of the Lorentz force density ${}^{(\mathrm{L})}f_\nu$, (6.29), as the source of the energy-momentum density ${}^{(\mathrm{F})}T_{\mu\nu}$ of the gauge field, now becomes immediately evident: for a closed system of matter and gauge fields the total energy-momentum as the sum of the gauge part ${}^{(\mathrm{F})}T_{\mu\nu}$ and matter part ${}^{(\mathrm{M})}T_{\mu\nu}$ must be conserved; i.e.,

$$\nabla^\mu({}^{(\mathrm{F})}T_{\mu\nu} + {}^{(\mathrm{M})}T_{\mu\nu}) \equiv 0 \tag{6.31}$$

But since the gauge part itself is not conserved individually [cf. (6.29)], this must be true also for the matter part,

$$\nabla^\mu\,{}^{(\mathrm{M})}T_{\mu\nu} = {}^{(\mathrm{L})}f_\nu = -\nabla^\mu\,{}^{(\mathrm{F})}T_{\mu\nu} \tag{6.32}$$

in order that the total system can be closed. These results may suggest that it is just the generalized Lorentz force ${}^{(\mathrm{L})}f_\nu$ which is responsible for the energy-momentum exchange between both subsystems. However, we shall readily see that this conjecture is not completely correct and the reason is that the Lorentz force cannot describe those exchange processes which are due to the materialization of gauge field energy (particle creation). In fact the Lorentz force ${}^{(\mathrm{L})}f_\nu$ does leave invariant the rest mass of the material subsystem, as we shall readily see by a closer inspection of its energy-momentum density ${}^{(\mathrm{M})}T_{\mu\nu}$ [see also the discussion of the classical equation (5.15)].

7. ENERGY-MOMENTUM DENSITY FOR MATTER

Besides the current (5.2a)–(5.2b) there emerges a further extrinsic density (i.e., observable) of matter and this refers to the energy-momentum content ${}^{(\mathrm{M})}T_{\mu\nu}$. Thus we have to specify now the energy-momentum density ${}^{(\mathrm{M})}T_{\mu\nu}$ of the *nonlinear* field system. As usual, the corresponding density is traced back to the "*energy-momentum operator*" ${}^{(\mathrm{M})}\mathcal{T}_{\mu\nu}$, i.e., we put

$${}^{(\mathrm{M})}T_{\mu\nu} = \mathrm{tr}(\mathcal{J}\cdot{}^{(\mathrm{M})}\mathcal{T}_{\mu\nu}) \tag{7.1a}$$

or, if a wave function $\psi(x)$ exists,

$${}^{(\mathrm{M})}T_{\mu\nu} = \bar{\psi}\cdot{}^{(\mathrm{M})}\mathcal{T}_{\mu\nu}\cdot\psi \tag{7.1b}$$

The operator ${}^{(M)}\mathcal{T}_{\mu\nu}$ has already been specified as (Sorg, n.d.)

$$ {}^{(M)}\mathcal{T}_{\mu\nu} = \frac{1}{2Mc^2}\,[\overline{\mathcal{H}}_\mu \cdot \mathcal{H}_\nu + \overline{\mathcal{H}}_\nu \cdot \mathcal{H}_\mu - g_{\mu\nu}(\overline{\mathcal{H}}^\lambda \cdot \mathcal{H}_\lambda - \mathcal{Y})] \tag{7.2} $$

where the new operator $\mathcal{Y}$ ($\rightsquigarrow$ "potential operator") still has to be determined. Clearly it is related to the mass operator $\mathcal{X}$ and therefore is also to be considered as a (matrix) function of the intensity matrix $\mathcal{J}$: $\mathcal{Y} = \mathcal{Y}(\mathcal{J})$. But for the linear theory, when $\mathcal{X}$ essentially simplifies to unity, (3.7), the corresponding $\mathcal{Y}$ is also reduced to

$$ \mathcal{Y} \Rightarrow \mathcal{Y}_{\text{lin}} = (\hbar c)^2 \mathcal{X}_{\text{lin}} = (Mc^2)^2 \cdot \mathbf{1} \tag{7.3} $$

Further information about the operator $\mathcal{Y}$ is obtained by considering the "*force density*" f_ν, which is the source of energy-momentum as usual,

$$ \nabla^\mu\, {}^{(M)}T_{\mu\nu} = f_\nu \tag{7.4} $$

The force density f_ν is again written in terms of the force operator $\mathfrak{f}_\nu$ as

$$ f_\nu = \text{tr}(\mathcal{J} \cdot \mathfrak{f}_\nu) \tag{7.5a} $$

or

$$ f_\nu = \overline{\psi} \cdot \mathfrak{f}_\nu \cdot \psi \tag{7.5b} $$

The desired link of the mass and potential operators $\mathcal{X}$, $\mathcal{Y}$ is now obtained by demanding that the force density f_ν be nothing else than the well-known Lorentz force (or its non-Abelian generalizations). Thus, in order to transcribe this demand into some equation connecting $\mathcal{X}$ and $\mathcal{Y}$, we look for the source of the energy-momentum density ${}^{(M)}T_{\mu\nu}$, (7.1a), and find by means of the Hamiltonian dynamics (3.4)–(3.5),

$$ \begin{aligned} \nabla^\mu\, {}^{(M)}T_{\mu\nu} &= \text{tr}\{\mathcal{J}(\mathcal{D}^\mu\, {}^{(M)}\mathcal{T}_{\mu\nu} + [\overline{\mathcal{H}}^\mu \cdot {}^{(M)}\mathcal{T}_{\mu\nu} - {}^{(M)}\mathcal{T}_{\mu\nu} \cdot \mathcal{H}^\mu])\} \\ &= \frac{i\hbar}{2Mc}\,\text{tr}(\mathcal{J}\{\overline{\mathcal{H}}^\mu \cdot \mathcal{F}_{\mu\nu} + \mathcal{F}_{\mu\nu} \cdot \mathcal{H}^\mu\}) \\ &\quad + \frac{\hbar}{2Mc}\,\text{tr}(\mathcal{J} \cdot \{\mathcal{G} \cdot \mathcal{H}_\nu + \overline{\mathcal{H}}_\nu \cdot \mathcal{G}\}) + \frac{1}{2Mc^2}\,\text{tr}\{\mathcal{J} \cdot (\mathcal{D}_\nu \mathcal{Y} \\ &\quad + \frac{i}{\hbar c}\,[\overline{\mathcal{H}}_\nu \cdot (\mathcal{Y} - (\hbar c)^2 \mathcal{X}) - (\mathcal{Y} - (\hbar c)^2 \mathcal{X})\mathcal{H}_\nu])\} \end{aligned} \tag{7.6} $$

Comparing this result to the expected form (7.4), we see that the general force density f_ν will consist of at least two parts,

$$ f_\nu = {}^{(L)}f_\nu + {}^{(C)}f_\nu \tag{7.7} $$

Here, the Lorentz force density ${}^{(L)}f_\nu$ is given by

$$ {}^{(L)}f_\nu = \frac{i\hbar}{2Mc}\,\mathrm{tr}(\mathcal{J}\cdot[\overline{\mathcal{H}}^\mu\cdot\mathcal{F}_{\mu\nu} + \mathcal{F}_{\mu\nu}\cdot\mathcal{H}^\mu]) \tag{7.8} $$

and similarly the conversion force density ${}^{(C)}f_\nu$ is defined through

$$ {}^{(C)}f_\nu = \frac{\hbar}{2Mc}\,\mathrm{tr}(\mathcal{J}\cdot[\mathcal{G}\cdot\mathcal{H}_\nu + \overline{\mathcal{H}}_\nu\cdot\mathcal{G}]) \tag{7.9} $$

Properly speaking, there emerges also a third force term on the right-hand side of the source equation (7.6), containing the $\mathcal{X}$ and $\mathcal{Y}$ operators, but this part will be required to vanish in order to get the desired link between the mass operator $\mathcal{X}$ and the potential operator $\mathcal{Y}$. In what follows we shall gain some further insight into RST by a closer inspection of any one of the three force terms.

First consider the convection force ${}^{(C)}\mathbf{f}$, (7.9). This part of the total force f_ν, (7.7), contains the convertor $\mathcal{G}$ which is responsible for matter creation and annihilation, i.e., for a change of the rest mass of the physical system (Sorg, n.d.). Such a change of the rest mass naturally leads to a kind of inertial force, which is best seen by considering the equations of motion for a point particle (5.15). There the classical four-momentum p_μ of the particle may be rewritten as the product of its rest mass M and four-velocity u_μ,

$$ p_\mu = Mcu_\mu, \qquad u^\mu = \frac{dx^\mu}{d\tau}, \quad u^\mu u_\mu = 1, \quad d\tau = \sqrt{g_{\mu\nu}dx^\mu dx^\nu} \tag{7.10} $$

Thus, the rate of the change of p_μ is given by two terms

$$ \frac{dp_\mu}{d\tau} = c\left(\frac{dM}{d\tau}\right)u_\mu + Mc\,\frac{du_\mu}{d\tau} \tag{7.11} $$

where the first one refers to the change of rest mass M and the second one to the four-acceleration $du_\mu/d\tau$. If only the Lorentz force is acting upon the particle, just as shown in equation (5.15), the rest mass must be a constant,

$$ \frac{d}{d\tau}(Mc)^2 \equiv \frac{d}{d\tau}(p^\mu p_\mu) = 0 \tag{7.12} $$

[hint: multiply the classical equation of motion (5.15) by p^ν]. On the other hand, when the rest mass of the physical system is changed, there must be added a conversion force term to the Lorentz contribution on the right of (5.15) for the sake of consistency. The same reasoning also applies to the source of energy-momentum density ${}^{(M)}T_{\mu\nu}$, (7.4), in our present fluid-dynamic approach to quantum mechanics, where the additional conversion force ${}^{(C)}f_\nu$ has already been identified in equation (7.9).

Next, the rest-mass-conserving Lorentz force ${}^{(L)}f_\nu$, (7.8), has to be discussed. Introducing here the decomposition of the field strength $\mathcal{F}_{\mu\nu}$ (6.5a), yields for the Lorentz force operator in terms of the velocity operator $v_{a\mu}$, (6.12),

$$
\begin{aligned}
{}^{(L)}\mathfrak{f}_\nu &= \frac{i\hbar}{2Mc}\left(\mathcal{H}^\mu \cdot \mathcal{F}_{\mu\nu} + \mathcal{F}_{\mu\nu} \cdot \mathcal{H}^\mu\right) \qquad (7.13)\\
&= \hbar c F_{\alpha\mu\nu} v^{a\mu}
\end{aligned}
$$

Obviously this is nothing else than the operator generalization of the classical Lorentz force on the right-hand side of equation (5.15). As a consequence, the Lorentz force density ${}^{(L)}f_\nu$, (7.8), becomes, in terms of the gauge currents $j_{a\mu}$, (6.9),

$$
{}^{(L)}f_\nu = \hbar c F_{a\mu\nu} j^{a\mu} \qquad (7.14)
$$

Clearly this result is just what one expects to be the non-Abelian generalization of the original Lorentz force in ordinary Maxwellian electromagnetism. The latter case is recovered here from our general result (7.8) by means of the Abelian substitutions (6.15a)–(6.15c)

$$
{}^{(L)}f_\nu \Rightarrow \frac{1}{c} F_{\mu\nu}\, {}^{(el)}j^\mu \equiv \frac{e}{Mc} F_{\mu\nu}\, {}^{(C)}j^\mu \qquad (7.15)
$$

Obviously, this is just the fluid-dynamic counterpart of the right-hand side of the classical equation of motion (5.15) and thus the original Bohm hypothesis receives support from RST according to which the energy-momentum content of a localized quantum system follows the classical path (= integral line of mass current) to lowest order of approximation [for relativistic quantum corrections to the classical path see Mattes and Sorg (1995)].

Finally, the third force term on the right of the source equation (7.6) has to be considered. This term is due to the presence of the potential operator $\mathcal{Y}$ in the energy-momentum operator ${}^{(M)}\mathcal{T}_{\mu\nu}$, (7.2), and therefore gives rise to the corresponding potential V in the density ${}^{(M)}T_{\mu\nu}$, (7.1a)–(7.1b), i.e., we put

$$
{}^{(Y)}T_{\mu\nu} = \frac{1}{2Mc^2} g_{\mu\nu} \cdot \mathrm{tr}(\mathcal{J} \cdot \mathcal{Y}) := V g_{\mu\nu} \qquad (7.16a)
$$

$$
V = \frac{1}{2Mc^2} \mathrm{tr}(\mathcal{J} \cdot \mathcal{Y}) \qquad (7.16b)
$$

For instance, if the mass operator $\mathcal{X}$ is the sum of two terms [as in equation (4.6)], then $\mathcal{Y}$ will also be additive,

$$\mathcal{Y}(\mathcal{J}) = \mathcal{Y}_H(\mathcal{J}) + \mathcal{Y}_F(\mathcal{J}) \tag{7.17}$$

which carries over to the potential V, (4.7), i.e.,

$$V = V_H + V_F \tag{7.18}$$

More concretely, the Fierz potential V_F, (4.4), due to the hitherto unknown mass operator $\mathcal{X}_F$ is found from the general relationship (7.16b) as

$$V_F = \frac{1}{2Mc^2}\,\mathrm{tr}(\mathcal{J}\cdot\mathcal{Y}_F) \tag{7.19}$$

Identifying therefore this expression with the previous choice (4.4) for V_F, we readily obtain the corresponding potential operator $\mathcal{Y}_F$ as

$$\begin{aligned}\mathcal{Y}_F(\rho, \mathcal{J}) &= 2(Mc^2)^2 a_F^2 \mathcal{J}^{-1}(\mathcal{D}_F)^2 \\ &= 2(Mc^2)^2 a_F^2(\mathcal{J}^3 - 2\rho\mathcal{J}^2 + \rho^2\mathcal{J})\end{aligned} \tag{7.20}$$

However, the inverse process of finding the Higgs potential V_H associated to the Higgs mass operator $\mathcal{X}_H$, (3.11), is not yet possible because we do not know how to compute the potential operator $\mathcal{Y}$ in terms of the mass operator $\mathcal{X}$ (and vice versa). Similarly, up to now we cannot compute the Fierz mass operator $\mathcal{X}_F$ in terms of its potential operator $\mathcal{Y}_F$, (7.20). In order to find the missing link between $\mathcal{X}$ and $\mathcal{Y}$, we adopt the postulate that matter does not feel any further force besides the Lorentz force ${}^{(L)}f_\nu$, (7.8), and conversion force ${}^{(C)}f_\nu$, (7.9). As a consequence of this postulate, the third force term on the right of the source equation (7.6) must vanish:

$$0 \overset{!}{=} \frac{1}{2Mc^2}\,\mathrm{tr}\left\{\mathcal{J}\cdot\left(\mathcal{D}_\nu\mathcal{Y} + \frac{i}{\hbar c}\left[\overline{\mathcal{H}}_\nu\cdot(\mathcal{Y} - (\hbar c)^2\mathcal{X}) - (\mathcal{Y} - (\hbar c)^2\mathcal{X})\cdot\mathcal{H}_\nu\right]\right)\right\} \tag{7.21}$$

Since the potential operator $\mathcal{Y}$ has been adopted to be a (matrix) function of the intensity matrix $\mathcal{J}$, cf. (7.20), the derivative of $\mathcal{Y}(\mathcal{J})$ in equation (7.21) can be traced back to the derivative of $\mathcal{J}$, which itself may be taken from its equation of motion (3.1). Thus, the postulate (7.21) establishes the following link between $\mathcal{X}$ and $\mathcal{Y}$:

$$\mathcal{Y}(\mathcal{J}) + \mathcal{J}\cdot\frac{\partial\mathcal{Y}}{\partial\mathcal{J}} + \mathbf{1}\cdot\mathrm{tr}\left(\mathcal{J}\cdot\frac{\partial\mathcal{Y}}{\partial\rho}\right) = (\hbar c)^2\mathcal{X} \tag{7.22}$$

Here, $\mathcal{Y}(\mathcal{J}, \rho)$ is understood as an analytic matrix function of $\mathcal{J}$ [e.g., the polynomial (7.20)].

For instance, applying the present result (7.22) to the Fierz case (7.20) readily yields for the Fierz mass operator $\mathscr{X}_F(2)$ in two dimensions ($N = 2$)

$$\mathscr{X}_F(2) = \left(\frac{2Mc}{\hbar}\right)^2 a_F^9 \left[(\rho \cdot \mathbf{1} - 2\mathscr{F})\mathscr{D}_F + \frac{1}{4}\, \rho \cdot \Delta_F^{(2)} \cdot \mathbf{1} \right]$$

$$= \left(\frac{Mc}{\hbar}\right)^2 a_F^9 \cdot \Delta_F^{(2)} \cdot (\rho \cdot \mathbf{1} + s_j \sigma^j) \tag{7.23}$$

i.e., the Fierz mass operator $\mathscr{X}_F$ vanishes identically for a pure state, where $\mathscr{D}_F = 0$ and thus *the Fierz potential can be felt only by a mixture*! Concerning the inverse problem of determining the potential V, (7.16b) for given mass operator $\mathscr{X}$, we may consider the Higgs case $\mathscr{X}_H$, (3.11), and again find from (7.22) the associated potential operator $\mathscr{Y}_H$ as

$$\mathscr{Y}_H = 2(Mc^2)^2(a_H^3 \mathscr{F} - \mathbf{1}) \tag{7.24}$$

Consequently, the desired Higgs potential (7.16b) becomes

$$V_H^{(2)} \Rightarrow Mc^2 a_H^3 \left[\left(\rho - \frac{1}{2a_H^3} \right)^2 - \frac{1}{2}(\rho^2 - s^2) \right] - \frac{Mc^2}{4a_H^3} \tag{7.25}$$

Including as usual some cosmological term

$${}^{(0)}T_{\mu\nu} = \frac{Mc^2}{4a_H^3}\, g_{\mu\nu} \tag{7.26}$$

into the energy-momentum tensor ${}^{(M)}T_{\mu\nu}$, (7.1a)–(7.1b), yields finally for the Higgs potential

$$V_H^{(2)}(\rho, s) = Mc^2 a_H^3 \left[\left(\rho - \frac{1}{2a_H^3} \right)^2 - \frac{1}{2}(\rho^2 - s^2) \right] \tag{7.27}$$

For pure states ($\rho \equiv s$), the Higgs potential V_H is nonnegative and is minimized by the nontrivial vacuum density $\rho_e = 1/(2a_H^3)$; however, the full potential V_H, (7.27), can become negative for a mixture ($\rho > |s|$), which indicates that pure states may decay into mixtures in order to minimize the *Higgs* energy [in contrast to the positive *Fierz* energy, (4.5)].

8. TRANSITIONS BETWEEN MIXTURES AND PURE STATES

According to the *nonunitary* space-time evolution of the intensity matrix $\mathscr{F}$ (or wave function ψ, respectively), it may become possible that the physical system is a mixture in one region of space-time but is in a pure state in some other region. In particular, such effects may occur with the progression of

"time" (to be defined suitably), i.e., we can consider systems which have been mixtures in the past and will become pure states in the distant future. Clearly, such a behavior would not be possible in ordinary quantum mechanics due to the postulate of strictly unitary time development of *closed* systems. But in RST, the tendency of minimizing the Fierz potential V_F, (4.5), drives the physical system toward becoming a pure state. On the other hand, the Higgs energy V_H, (7.27), can be decreased by leaving the pure state and passing over to a mixture. Consequently, it will be interesting to inspect which of these tendencies will dominate and therefore will determine the nature of the physical state. Subsequently we shall study these questions for fiber dimensions $N = 2$, but the results are expected to hold for any N.

In order to get some preliminary feeling for what happens, it may be instructive to consider the four-dimensional configuration space D_4 of the densities ρ, s^j (2.4a)–(2.4b) (see Fig. 1). This space $D_4 = \{\rho, s^j\}$ is the union of the sets $D_4 = D_- \cup D_+ \cup D_0 \cup C_F$ and carries a pseudo-Euclidean structure with the Fierz deviation $\Delta_F^{(2)}$, (4.3), as the Minkowskian metric. The characteristic feature of such a D_4 is its subdivision into four submanifolds:

(i) The "*Fierz cone*" $C_F = C_F^{(+)} \cup C_F^{(-)}$ as the union of the "*forward cone*" $C_F^{(+)}$ and the "*backward cone*" $C_F^{(-)}$,

$$C_F^{(+)}: \quad \rho = |\mathbf{s}| \tag{8.1a}$$

$$C_F^{(-)}: \quad \rho = -|\mathbf{s}|$$

(ii) The interior D_+ of $C_F^{(+)}$,

$$D_+: \quad \rho > |\mathbf{s}| \tag{8.1b}$$

(iii) The interior D_- of $C_F^{(-)}$,

$$D_-: \quad \rho < -|\mathbf{s}| \tag{8.1c}$$

(iv) The outside D_0 of C_F,

$$D_0: \quad -|\mathbf{s}| < \rho < +|\mathbf{s}| \tag{8.1d}$$

The question to be studied now is whether the Fierz cone C_F (or even $C_F^{(+)}$) is an "*attractor*" in the sense that any initial density configuration $\{\rho_{in}, s^j_{in}\}$ develops toward C_F (or even $C_F^{(+)}$). If such a configuration path $\rho = \rho(\theta)$, $s^j = s^j(\theta)$ tends to terminate at some point of $C_F^{(+)}$, one will finally find a pure state whose scalar density $\rho = \mathrm{tr}\, \mathcal{S}$ is trivially positive ($\rightsquigarrow \rho \equiv \bar{\psi} \cdot \psi$). Since the scalar ρ enters also the physical densities, e.g., the mass current ${}^{(C)}j_\mu$, (5.21), the question of its positivity ($\rho \gtrless 0$?) is found to be of some significance! Moreover, one wants to know whether the configuration paths can also traverse the Fierz cone C_F or whether they must always stay within their initial compartments $D_\pm$, D_0 (as is the case for the free-particle world

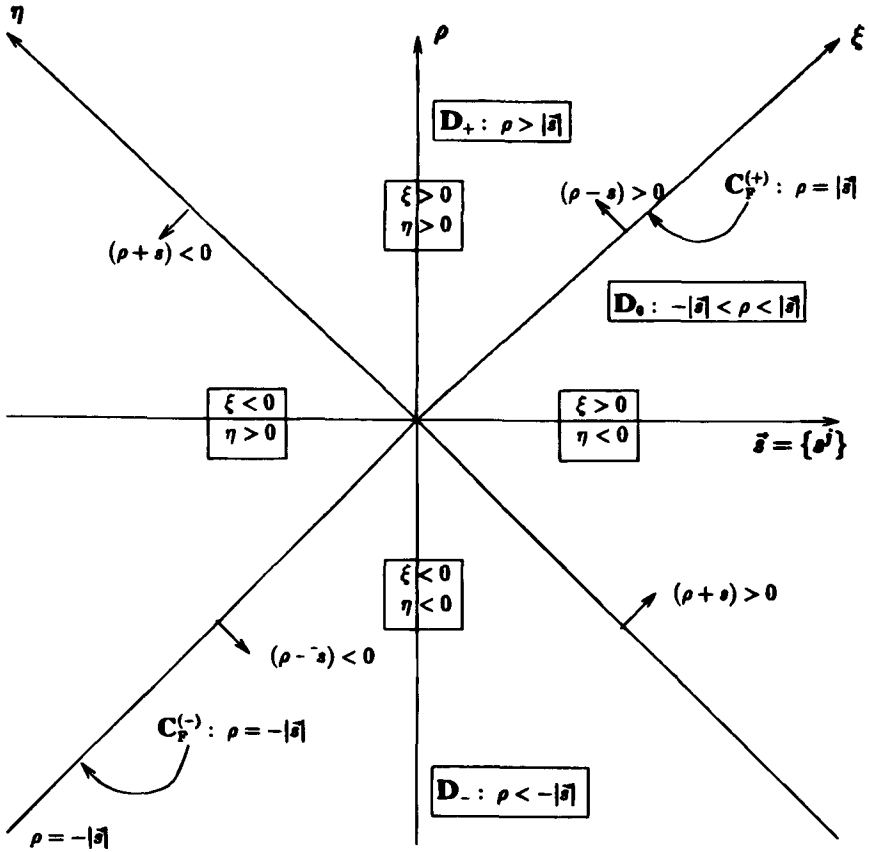

Fig. 1. Density configuration space.

lines in special relativity, which remain either inside or outside the corresponding light cone).

First, let us consider the problem of cone traversion. From the field equations (6.24a)–(6.24b) for the densities $\{\rho, s^j\}$ one easily deduces the field equation for the Fierz deviation $\Delta_F^{(2)}$, (4.3),

$$\partial_\mu \Delta_F^{(2)} = \partial_\mu(\rho^2 - s^2) = 4L_\mu(\rho^2 - s^2) \tag{8.2}$$

On the other hand, the Hamiltonian coefficient L_μ, (5.18b), can be shown to be the logarithmic gradient of some "*amplitude field*" $L(x)$, i.e.,

$$L_\mu = \frac{\partial_\mu L}{L} \tag{8.3}$$

Indeed, this can be readily verified by writing down the integrability condition (3.4) separately for the kinetic field $\mathcal{H}_\mu$, (5.18a),

$$\nabla_\mu K_\nu - \nabla_\nu K_\mu = F_{\mu\nu} \tag{8.4a}$$

$$D_\mu K_{j\nu} - D_\nu K_{j\mu} - 2\epsilon_j^{kl}(K_{k\mu}K_{l\nu} - L_{k\mu}L_{l\nu}) = \tfrac{1}{2}F_{j\mu\nu} \tag{8.4b}$$

and similarly for the localization field $\mathcal{L}_\mu$, (5.18b),

$$\nabla_\mu L_\nu - \nabla_\nu L_\mu = 0 \tag{8.5a}$$

$$D_\mu L_{j\nu} - D_\nu L_{j\mu} - 2\epsilon_j^{kl}(K_{k\mu}L_{l\nu} - K_{k\nu}L_{l\mu}) = 0 \tag{8.5b}$$

Here, equation (8.5a) immediately implies the desired gradient relationship (8.3). But once this gradient relation has been established, the field equation for the Fierz deviation $\Delta_F^{(2)}$, (8.2), is readily integrated to yield

$$\Delta_F^{(2)}(x) = \Delta_{F,\text{in}}^{(2)}\left(\frac{L(x)}{L_{\text{in}}}\right)^4 \tag{8.6}$$

This important result immediately provides the answer to the question raised above: if the density configuration initially is *inside* the Fierz cone [$\Delta_{F,\text{in}}^{(2)} := \Delta_F^{(2)}(x_{\text{in}}) > 0$], then it *remains* inside ($D_+ \cup D_-$) for all events x of space-time which can be connected continuously with the initial event x_{in} [the zeros of $L(x)$ are discussed below]. Evidently, an analogous result holds also for the outside region D_0. Conversely, a pure state always must extend over the whole (connected component of) space-time ($\Delta_{F,\text{in}}^{(2)} = 0$).

However, these results do *not* forbid that the density configuration *asymptotically*, i.e., for $\theta \to \infty$, approaches some point of the Fierz cone (off the vertex $\rho = s = 0$), i.e., the physical system asymptotically would become a pure state along some *infinite* path in space-time. In order to see this more clearly, we once more start from the field equations (6.24a)–(6.24b) for the scalar fields ρ and s ($:= \sqrt{-s_j s^j}$)

$$\partial_\mu \rho = 2(\rho L_\mu + s\overset{\cap}{L}_\mu) \tag{8.7a}$$

$$\partial_\mu s = 2(\rho\overset{\cap}{L}_\mu + sL_\mu) \tag{8.7b}$$

where we have introduced the logarithmic field $\overset{\cap}{L}_\mu$ and associated amplitude field $\overset{\cap}{L}(x)$ quite analogously to the preceding case (8.3),

$$\overset{\cap}{L}_\mu = \hat{s}^j L_{j\mu} = \frac{\partial_\mu \overset{\cap}{L}}{\overset{\cap}{L}} \tag{8.8}$$

Adding equations (8.7a)–(8.7b) immediately yields the sum of density scalars in terms of the amplitude fields L, $\overset{\cap}{L}$ as

$$(\rho + s)(x) = (\rho + s)_{\text{in}}\left(\frac{L(x)}{L_{\text{in}}}\right)^2\left(\frac{\overset{\cap}{L}(x)}{\overset{\cap}{L}_{\text{in}}}\right)^2 \tag{8.9}$$

In a similar way, the subtraction of equations (8.7a)–(8.7b) leads to

$$(\rho - s)(x) = (\rho - s)_{\text{in}} \left(\frac{L(x)}{L_{\text{in}}}\right)^2 \left(\frac{\overset{\cap}{L}_{\text{in}}}{\overset{\cap}{L}(x)}\right)^2 \tag{8.10}$$

Of course, both results (8.9) and (8.10) are in agreement with the previous form (8.6) for the Fierz deviation $\Delta_F^{(2)} \equiv (\rho + s)(\rho - s) = \rho^2 - s^2$. But now observe that it may be well possible that along some infinite path in space-time, both scalars ρ and s asymptotically tend to the same value $\rho \Rightarrow s$ ($\neq 0$) if only the first amplitude field $L(x)$ tends to zero [$L(x) \Rightarrow 0$] and the second one tends to infinity [$\overset{\cap}{L}(x) \Rightarrow \infty$] so that their product remains finite

$$\left(\frac{L(x)}{L_{\text{in}}}\right)^2 \left(\frac{\overset{\cap}{L}(x)}{\overset{\cap}{L}_{\text{in}}}\right)^2 \Rightarrow \frac{2\rho_\infty(x)}{\rho_{\text{in}} + s_{\text{in}}} = \frac{2s_\infty(x)}{\rho_{\text{in}} + s_{\text{in}}} \tag{8.11}$$

Indeed, with these presumptions equation (8.9)–(8.10) predict the emergence of a pure state ($\rho = s$) and the Fierz deviation $\Delta_F^{(2)}$, (8.6), will vanish as required. Thus, the transition of a mixture into a pure state becomes possible from the purely *kinematical* point of view.

But now the question arises whether these transitions are admitted also from the *dynamical* viewpoint. Here, a rigorous proof of their existence would surely be very difficult for the most general situation; therefore we want to consider a very special example which, however, admits certain conclusions concerning the general result. Our choice of an example refers to a homogeneous and isotropic field configuration over an expanding RW universe. The reason is here that the high symmetry of such a field configuration reduces the space-time evolution to a pure problem of time development. This arrangement simplifies all the preceding field equations into ordinary differential equations in terms of cosmic time θ. Consequently, we shall specify the initial values of the dynamical variables at the initial time θ_{in} and then follow their time evolution in order to see whether the mixture asymptotically becomes a pure state in the distant future ($\theta \to \infty$).

For an investigation of this kind, we first have to specify the dynamical equations in their RW-symmetric form, where all scalar fields become exclusive functions of cosmic time θ and vector fields become proportional to the Hubble flow b_μ (Ochs and Sorg, n.d.-a). For instance, the results (8.9)–(8.10) would now read

$$(\rho + s)(\theta) = (\rho + s)_{\text{in}} \left(\frac{L(\theta)}{L_{\text{in}}}\right)^2 \left(\frac{\overset{\cap}{L}(\theta)}{\overset{\cap}{L}_{\text{in}}}\right)^2 \tag{8.12a}$$

$$(\rho - s)(\theta) = (\rho - s)_{\text{in}}\left(\frac{L(\theta)}{L_{\text{in}}}\right)^2\left(\frac{\overset{\cap}{L}_{\text{in}}}{\overset{\cap}{L}(\theta)}\right)^2 \tag{8.12b}$$

Similarly, the kinetic and localization coefficients (5.18a)–(5.18b) must look as follows:

$$K_\mu = Kb_\mu \tag{8.13a}$$

$$K_{j\mu} = K_j b_\mu \tag{8.13b}$$

$$L_\mu = \Lambda b_\mu \tag{8.13c}$$

$$L_{j\mu} = L_j b_\mu \tag{8.13d}$$

where the scalar prefactors are again homogeneous scalar fields [i.e., $K = K(\theta)$, etc.]. Next observe that the intrinsic densities s^j, (2.4b), define a preferred direction in the corresponding fiber space, and for the subsequent discussion it will be helpful to decompose all $SO(3)$ gauge objects with respect to that direction, i.e., we put

$$K_j = -\hat{s}_j\overset{\cap}{K} + {}^{(\perp)}K_j \tag{8.14a}$$

$$L_j = -\hat{s}_j\overset{\cap}{\Lambda} + {}^{(\perp)}L_j \tag{8.14b}$$

Thus, $\overset{\cap}{K}$ and $\overset{\cap}{\Lambda}$ are the projectors of the kinetic and localization coefficients $\{K_j, L_j\}$ onto the preferred direction

$$\overset{\cap}{K} = \hat{s}^j K_j \tag{8.15a}$$

$$\overset{\cap}{\Lambda} = \hat{s}^j L_j \tag{8.15b}$$

In this way, the preferred direction separates the variables into two subsets, namely the "*longitudinal*" objects $\{K, \overset{\cap}{K}, \Lambda, \overset{\cap}{\Lambda}\}$ and the "*transverse*" quantities $\{K_\perp^2, L_\perp^2, S_\perp, N_\perp\}$ to be defined as follows:

$$K_\perp^2 := -{}^{(\perp)}K^j \cdot {}^{(\perp)}K_j \tag{8.16a}$$

$$L_\perp^2 := -{}^{(\perp)}L^j \cdot {}^{(\perp)}L_j \tag{8.16b}$$

$$S_\perp := {}^{(\perp)}L^j \cdot {}^{(\perp)}K_j \tag{8.16c}$$

$$N_\perp := \epsilon_j^{kl} K_k \hat{s}_l L^j \tag{8.16d}$$

After these purely kinematical adaptations to the RW symmetry, we can

now turn to the corresponding field equations. First, the density dynamics (8.7a)–(8.7b) is rewritten as follows:

$$b^\mu \partial_\mu \rho := \dot{\rho} = 2(\rho \cdot \Lambda + s \cdot \overset{\cap}{\Lambda}) \tag{8.17a}$$

$$b^\mu \partial_\mu s := \dot{s} = 2(\rho \cdot \overset{\cap}{\Lambda} + s \cdot \Lambda) \tag{8.17b}$$

Moreover, the gradient relations (8.3) and (8.8) read now

$$\Lambda = \dot{L}/L \tag{8.18a}$$

$$\overset{\cap}{\Lambda} = \dot{\overset{\cap}{L}}/\overset{\cap}{L} \tag{8.18b}$$

and consequently the relationships (8.12a)–(8.12b) between densities $\{\rho, s\}$ and amplitude fields $\{L, \overset{\cap}{L}\}$ are immediately recognized as the solutions of the density dynamics (8.17a)–(8.17b). Next, the system of conservation equations (6.25a)–(6.26b) yields the equations of motion for the longitudinal variables

$$\dot{\Lambda} + (3H + \Lambda)\Lambda + \overset{\cap}{\Lambda}^2 - \overset{\cap}{K}^2 - K^2 + X = K_\perp^2 - L_\perp^2 \tag{8.19a}$$

$$\dot{\overset{\cap}{\Lambda}} + (3H + 2\Lambda)\overset{\cap}{\Lambda} - 2K \cdot \overset{\cap}{K} + \overset{\cap}{X} = 2\left(N_\perp + \frac{\rho}{s} L_\perp^2\right) \tag{8.19b}$$

$$\dot{K} + (3H + 2\Lambda)K + 2\overset{\cap}{\Lambda} \cdot \overset{\cap}{K} - \frac{1}{2} G = 2S_\perp \tag{8.19c}$$

$$\dot{\overset{\cap}{K}} + (3H + 2\Lambda)\overset{\cap}{K} + 2\overset{\cap}{\Lambda} \cdot K + \frac{1}{2} G = -2\frac{\rho}{s} S_\perp \tag{8.19d}$$

and for the transverse variables

$$\dot{K}_\perp^2 = -2(3H + 2\Lambda)K_\perp^2 + 4\left(K + \frac{\rho}{s}\overset{\cap}{K}\right)S_\perp \tag{8.20a}$$

$$\dot{L}_\perp^2 = -2\left(3H + 2\Lambda + 2\frac{\rho}{s}\overset{\cap}{\Lambda}\right)L_\perp^2 - 4(K \cdot S_\perp + \overset{\cap}{\Lambda} \cdot N_\perp) + 2({}^{(\perp)}L^j \cdot X_j) \tag{8.20b}$$

$$\dot{S}_\perp = -2\left(3H + 2\Lambda + \frac{\rho}{s}\overset{\cap}{\Lambda}\right)S_\perp + 2K(L_\perp^2 - K_\perp^2) + 2\overset{\cap}{K}\left(N_\perp + \frac{\rho}{s} L_\perp^2\right) - ({}^{(\perp)}K^j \cdot X_j) \tag{8.20c}$$

$$\dot{N}_\perp = -2\left(3H + 2\Lambda + \frac{\rho}{s}\hat{\Lambda}\right)\cdot N_\perp - 2(\hat{\Lambda}\cdot K_\perp^2 + \hat{K}\cdot S_\perp) - \epsilon_j^{kl}K_k\hat{s}_l X^j \quad (8.20d)$$

Observe that this somewhat complicated system lives over a RW universe with Hubble expansion rate H, but it can be simplified somewhat if we think of the mass operator $\mathscr{X}$

$$\mathscr{X} = X\cdot 1 + X_j\sigma^j \qquad (\hat{X} := \hat{s}^j X_j) \quad (8.21)$$

as composed of a Higgs part $\mathscr{X}_H$, (3.11), and a Fierz part, $\mathscr{X}_F$, (7.23), which both have no transverse component (i.e., ${}^{(\perp)}X_j \equiv 0$). Thus all terms containing the "space part" X_j of the mass operator can be omitted in the transverse dynamics (8.20a)–(8.20d). Moreover, since we want to follow the time evolution of a mixture, the conversion density G *must* be put equal to zero; see the remark below (5.22).

But now with the complete dynamics at hand, we can effectively attack the problem of asymptotic transitions into pure states. For that purpose, we will set up a closed differential equation for the trajectory $\{\rho = \rho(\theta), s^j = s^j(\theta)\}$ in the density configuration space (see Fig. 1), and from this differential equation we will recognize that the corresponding density trajectories can never terminate upon the Fierz cone representing the pure states. This means that a mixture can never become a pure state.

First observe that there exists a close relationship between the densities $\{\rho, s\}$ and the amplitude fields $\{L, \hat{L}\}$, which is most clearly expressed by the corresponding solutions (8.12a)–(8.12b). On the other hand, a similar relation between the kinetic fields $\{K, \hat{K}\}$ and the densities $\{\rho, s\}$ apparently does not exist. Consequently, it may seem somewhat hard to eliminate the kinetic fields for arriving at the desired closed equation for the densities. But in order to manage this problem, we refer to the results of Sorg (n.d.), where it is shown that in an expanding universe ($\rightsquigarrow$ positive expansion rate H) the transverse variables $\{K_\perp^2, L_\perp^2, S_\perp, N_\perp\}$ rapidly decay to zero and thus are clearly outlived by the longitudinal quantities $\{\Lambda, \hat{\Lambda}, K, \hat{K}\}$. For such a situation we can restrict ourselves to that late phase of the dynamical evolution where only the longitudinal variables are alive and then obey the simplified equations of motion (8.19a)–(8.19d),

$$\dot{\Lambda} + (3H + \Lambda)\Lambda + \hat{\Lambda}^2 - \hat{K}^2 - K^2 + X = 0 \quad (8.22a)$$

$$\dot{\hat{\Lambda}} + (3H + 2\Lambda)\hat{\Lambda} - 2K\cdot\hat{K} + \hat{X} = 0 \quad (8.22b)$$

$$\dot{K} + (3H + 2\Lambda)K + 2\hat{\Lambda}\cdot\hat{K} = 0 \quad (8.22c)$$

$$\dot{\overset{\frown}{K}} + (3H + 2\Lambda)\overset{\frown}{K} + 2\overset{\frown}{\Lambda}\cdot K = 0 \tag{8.22d}$$

The advantage of this truncated dynamical system is that it admits the desired elimination of the kinetic fields in a rather elegant way.

Indeed, adding equations (8.22c) and (8.22d) and observing the density dynamics (8.17a)–(8.17b) readily yields for the sum of the kinetic fields in terms of the densities

$$K(\theta) + \overset{\frown}{K}(\theta) = (K_{\text{in}} + \overset{\frown}{K}_{\text{in}})\left(\frac{\mathcal{R}_{\text{in}}}{\mathcal{R}}\right)^3 \frac{\rho_{\text{in}} + s_{\text{in}}}{\rho(\theta) + s(\theta)} \tag{8.23a}$$

Moreover, subtracting in place of adding yields

$$K(\theta) - \overset{\frown}{K}(\theta) = (K_{\text{in}} - \overset{\frown}{K}_{\text{in}})\left(\frac{\mathcal{R}_{\text{in}}}{\mathcal{R}}\right)^3 \frac{\rho_{\text{in}} - s_{\text{in}}}{\rho(\theta) - s(\theta)} \tag{8.23b}$$

This is the desired relation for the elimination of the kinetic fields in favor of the densities. But observe that, according to these results, the longitudinal kinetic fields must become singular on the Fierz cone C_F. As we shall readily see, this singular behavior on C_F is the crucial point for the possibility of transitions into the pure states.

In the next step we will add and subtract equations (8.22a) and (8.22b) in a similar way, but it is convenient to first introduce here some new notations: for the sum of the localization coefficient Λ, $\overset{\frown}{\Lambda}$ we put [cf. (8.17a)–(8.17b)]

$$\Lambda + \overset{\frown}{\Lambda} = \frac{1}{2}\frac{1}{\rho + s}\frac{d}{d\theta}(\rho + s) := \frac{\dot{\xi}}{\xi} \tag{8.24a}$$

$$\xi := \begin{cases} \sqrt{|\rho + s|}, & \rho + s > 0 \\ -\sqrt{|\rho + s|}, & \rho + s < 0 \end{cases}$$

and analogously for the difference

$$\Lambda - \overset{\frown}{\Lambda} = \frac{1}{2}\frac{1}{\rho - s}\frac{d}{d\theta}(\rho - s) := \frac{1}{\eta}\frac{d\eta}{d\theta} \tag{8.24b}$$

$$\eta := \begin{cases} \sqrt{|\rho - s|}, & \rho - s > 0 \\ -\sqrt{|\rho - s|}, & \rho - s < 0 \end{cases}$$

Furthermore, the relevant constituents X and $\overset{\frown}{X}$ of the mass operator $\mathcal{X}$, (8.21), are expressed in terms of the new variables ξ, η as

$$\xi \cdot (X + \overset{\cap}{X}) = \frac{\partial U_\infty(\xi)}{\partial \xi} \tag{8.25a}$$

$$\eta \cdot (X - \overset{\cap}{X}) = \frac{\partial U_\infty(\eta)}{\partial \eta} \tag{8.25b}$$

with the "*fictitious potential*" U_∞ being given by

$$U_\infty(\xi) = \left(\frac{Mc}{\hbar}\right)^2 \left(\pm \frac{1}{2} a_H^3 \xi^4 - \xi^2 \pm \frac{1}{4} a_F^9 \xi^4 \eta^4 \right) \tag{8.26a}$$

(upper sign: $\xi > 0$; lower sign: $\xi < 0$)

and

$$U_\infty(\eta) = \left(\frac{Mc}{\hbar}\right)^2 \left(\pm \frac{1}{2} a_H^3 \eta^4 - \eta^2 \pm \frac{1}{4} a_F^9 \xi^4 \eta^4 \right) \tag{8.26b}$$

(upper sign: $\eta > 0$; lower sign: $\eta < 0$)

Obviously, within the Fierz cone $D_+ \cup D_-$ it is possible to combine both partial potentials (8.26a)–(8.26b) into a single one [$U_\infty(\xi, \eta)$, say], namely

$$U_\infty(\xi, \eta) = \left(\frac{Mc}{\hbar}\right)^2 \left(\pm \frac{1}{2} a_H^3 (\xi^4 + \eta^4) - (\xi^2 + \eta^2) \pm \frac{1}{4} a_F^9 \xi^4 \eta^4 \right) \tag{8.27}$$

(upper sign: D_+; lower sign: D_-)

but outside the Fierz cone (i.e., in D_0) the introduction of such a total potential $U_\infty(\xi, \eta)$ is not possible.

Consequently, with these arrangements the addition and subtraction of the first two equations of the system (8.22a)–(8.22d) yields a Newtonian equation for the motion of a fictive mechanical point particle over density configuration space being reparametrized by the coordinates $\{\xi, \eta\}$, (8.24a)–(8.24b). In D_0 (i.e., outside the Fierz cone), this equation of motion reads

$$\frac{d^2\xi}{d\theta^2} + 3H \frac{d\xi}{d\theta} - (K_{in} + \overset{\cap}{K}_{in})^2 \left(\frac{\mathcal{R}_{in}}{\mathcal{R}}\right)^6 \frac{\xi_{in}^4}{\xi^3} = -\frac{\partial U_\infty(\xi)}{\partial \xi} \tag{8.28a}$$

$$\frac{d^2\eta}{d\theta^2} + 3H \frac{d\eta}{d\theta} - (K_{in} - \overset{\cap}{K}_{in})^2 \left(\frac{\mathcal{R}_{in}}{\mathcal{R}}\right)^6 \frac{\eta_{in}^4}{\eta^3} = -\frac{\partial U_\infty(\eta)}{\partial \eta} \tag{8.28b}$$

because in D_0 there does not exist the combined potential $U(\xi, \eta)$, (8.27). But within the Fierz cone $D_+ \cup D_-$, the motion of the fictive particle can be thought to take place under the action of the combined potential $U_\infty(\xi, \eta)$, to be substituted on the right-hand sides of (8.28a)–(8.28b) in place of

those partial potentials $U_\infty(\xi)$ and $U_\infty(\eta)$. Moreover, within that region $D_+ \cup D_-$ the kinetic terms on the left-hand sides of (8.28a)–(8.28b) can also be absorbed into some potential $U_K(\xi, \eta)$ (*"kinetic potential"*)

$$U_K(\xi, \eta) = \left(\frac{\mathcal{R}_{in}}{\mathcal{R}}\right)^6 \left[(K_{in} + \overset{\cap}{K}_{in})^2 \frac{\xi_{in}^4}{\xi^2} + (K_{in} - \overset{\cap}{K}_{in})^2 \frac{\eta_{in}^4}{\eta^2} \right] \tag{8.29}$$

so that the total potential $U(\xi, \eta)$ becomes

$$U(\xi, \eta) = U_K(\xi, \eta) + U_\infty(\xi, \eta) \tag{8.30a}$$

$$\lim_{\mathcal{R}\to\infty} U(\xi, \eta) = U_\infty(\xi, \eta) \tag{8.30b}$$

By use of this total potential $U(\xi, \eta)$, the Newtonian equations (8.28a)–(8.28b) simply read for the region $D_+ \cup D_-$

$$\frac{d^2\xi}{d\theta^2} + 3H\frac{d\xi}{d\theta} = -\frac{\partial U(\xi, \eta)}{\partial \xi} \tag{8.31a}$$

$$\frac{d^2\eta}{d\theta^2} + 3H\frac{d\xi}{d\theta} = -\frac{\partial U(\xi, \eta)}{\partial \eta} \tag{8.31b}$$

With these results at hand, we are now able to decide the question of the transitions from mixtures into pure states. Within the Newtonian point-particle approach such a transition would be equivalent to the termination of the particle's motion at the Fierz cone C_F. However, this is not possible, because the kinetic potential U_K, (8.29), becomes infinite on the Fierz cone C_F ($\xi = 0, \eta = 0$) and therefore the cone C_F acquires the repulsive properties of an impenetrable barrier for the particle. This repulsive property exists on both sides of the barrier (i.e., inside and outside of the Fierz cone C_F) and it persists even in the limit of an increasing extension of the universe ($\mathcal{R} \to \infty$). Since this repulsive effect is due to the *kinetic* potential U_K, it is an intrinsically dynamic effect and therefore completely independent of the presence of some *nonlinear potential* of the Higgs or Fierz type. For this reason, we expect that in RST *the mixtures quite generally cannot become pure states,* although the present results verify our expectation only for fiber dimension $N = 2$. Observe also that these conclusions are independent of whether the universe is expanding or not ($\rightsquigarrow H \geq 0$). The reason is that the expansion effect enters the mechanical equations (8.31a)–(8.31b) mainly in form of a friction term ($\sim H$) and the presence of friction cannot falsify the mechanical conclusions drawn above. Furthermore, the repulsive cone potential U_K remains present for any finite value $\mathcal{R}$ of the universe's size and consequently it does not help making the universe infinitely large ($\mathcal{R} \to \infty$). On the other hand, if we start with an exact pure state, then it will remain

a pure state for all future time, as predicted by equation (8.6). But if we start with some mixture arbitrarily close to a pure state, the repulsive potential U_K will cause the fictive particle to move off the Fierz cone, and correspondingly the (approximate) pure state will be transmuted into a true mixture. This transmutation will be irreversible if dissipation is included ($\rightsquigarrow$ expansion of the universe).

REFERENCES

Abbott, L. F. and Pi, S.-Y. (1986). *Inflationary Cosmology,* World Scientific, Singapore.

Crawford, J. P. (1985). *Journal of Mathematical Physics,* **26**, 1439.

Fierz, M. (1937), *Zeitschrift für Physik,* **104**, 553.

Ghirardi, G. C., Rimini, A., and Weber, T. (1976a). *Nuovo Cimento,* **31B**, 177.

Ghirardi, G. C., Rimini, A., and Weber, T. (1976b). *Nuovo Cimento,* **36B**, 97.

Huang, K. (1982). *Quarks, Leptons, and Gauge Fields,* World Scientific, Singapore.

Kolb, E. W., and Turner, M. S. (1990). *The Early Universe,* Addison-Wesley, Reading, Massachusetts.

Mattes, M., and Sorg, M. (1994). *Nuovo Cimento,* **109B**, 1097.

Mattes, M., and Sorg, M. (1995). *Nuovo Cimento,* **110B**, 1323.

Ochs, U., and Sorg, M. (n.d.-a). Cosmological solutions for the coupled Einstein–Yang–Mills–Higgs equations, *General Relativity and Gravitation,* to appear.

Ochs, U., and Sorg, M. (n.d.-b). Non-linear generalizations of the relativistic Schrödinger equations, *Zeitschrift für Naturforschung,* to appear.

Quigg, C. (1983). *Gauge Theories of the Strong, Weak, and Electromagnetic Interactions,* Benjamin/Cummings, Palo Alto, California.

Selleri, F. (1990). *Quantum Paradoxes and Physical Reality,* Kluwer, Dordrecht.

Sorg, M. (n.d.). Identities and conservation laws in relativistic Schrödinger theory, *Nuovo Cimento B,* to appear.

Weinberg, S. (1989). *Annals of Physics,* **194**, 336.

Wheeler, J. A., and Zurek, W. H. (1983). *Quantum Theory of Measurement,* Princeton University Press, Princeton, New Jersey.